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Eigenvectors of unitary ϱ-dilations

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Let T be a linear bounded operator on a Hilbert space H and ϱ a positive number. We say that U is a unitary ϱ-dilation of T, if U is a unitary operator on a Hilbert space $K \supset H$ and

$$T^n h = \varrho P U^n h \quad \text{for all} \quad h \in H \quad \text{and for} \quad n = 1, 2, \ldots,$$

where P (as always in the following) denotes the orthogonal projection of K onto H. Clearly U is a unitary ϱ-dilation of T if and only if U^{-1} is a unitary ϱ-dilation of T^*. U is called to be minimal, if

$$K = \bigvee_{n=-\infty}^{\infty} U^n H.$$

$\mathscr{C}_\varrho$ denotes the class of those operators which have unitary ϱ-dilations.

Unitary ϱ-dilations were introduced and operators of classes $\mathscr{C}_\varrho$ were characterized by SZ.-NAGY and FOIAŞ [6]. Spectral properties of unitary ϱ-dilations were studied in [1], [5], [3]. In this Note we prove two theorems about eigenvalues and eigenvectors, generalizations to the unitary ϱ-dilation case of facts known for the unitary 1-dilations ([7], Ch. 2, Proposition 6.1).

In what follows we fix a positive number ϱ, a minimal unitary ϱ-dilation U of T, and introduce the following notations:

$$(1) \qquad K_+ = \bigvee_{n=0}^{\infty} U^n H, \quad U_+ = U|K_+; \quad K_- = \bigvee_{n=0}^{\infty} U^{-n} H, \quad U_- = U^{-1}|K_-;$$

$$L_+ = K_+ \ominus U K_+, \quad L_- = K_- \ominus U^{-1} K_-;$$

$$(2) \qquad R_+ = \bigcap_{n=0}^{\infty} U^n K_+, \quad R_- = \bigcap_{n=0}^{\infty} U^{-n} K_-;$$

$$(3) \qquad R_0 = R_+ \cap R_-.$$

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Clearly, L_+ and L_- are wandering subspaces for the isometric operators U_+ and U_-, respectively. Moreover,

$$K_+ = \left(\bigoplus_{n=0}^{\infty} U^n L_+\right) \oplus R_+, \quad K_- = \left(\bigoplus_{n=0}^{\infty} U^{-n} L_-\right) \oplus R_-$$

are the corresponding Wold decompositions. By the minimality of U this implies that

$$K = \left(\bigoplus_{n=-\infty}^{\infty} U^n L_+\right) \oplus R_+, \quad K = \left(\bigoplus_{n=-\infty}^{\infty} U^{-n} L_-\right) \oplus R_-. \tag{4}$$

Finally we denote by P_+, P_-, and P_0 the orthogonal projections of K onto R_+, R_-, and R_0, respectively.

Theorem A. (Mlak [4]) $U^n T^{*n} h \rightharpoonup P_+ h, \quad U^{-n} T^n h \rightharpoonup P_- h \quad (\forall h \in H)$ *(weak convergence).*

Theorem 1. *If*

$$Ug = \varepsilon g \tag{5}$$

for some $g \in K$ and complex number ε, then

$$TPg = \varepsilon Pg, \quad T^* Pg = \bar{\varepsilon} Pg, \quad \textit{and} \quad g \in R_0.$$

Proof. Let Q_n denote the orthogonal projection of g onto $U^n L_+$ ($n=0, \pm 1, \pm 2, \ldots$), then by (4)

$$\sum_{n=-\infty}^{\infty} \|Q_n g\|^2 \leqq \|g\|^2, \quad UQ_n g = Q_{n+1} Ug, \tag{6}$$

and so by (5)

$$\|Q_n g\| = \|Q_{n+1} Ug\| = \|Q_{n+1} \varepsilon g\| = \|Q_{n+1} g\| \quad (n = 0, \pm 1, \pm 2, \ldots).$$

By (6) this implies $Q_n g = 0$ for all n, thus by (4)

$$g \in R_+. \tag{7}$$

For $n = 1, 2, \ldots$ and $h \in H$ we have

$$TP(U^n h) = \frac{1}{\varrho} T^{n+1} h = PU(U^n h).$$

Since by (1) and (2)

$$R_+ = \bigcap_{n=0}^{\infty} \bigvee_{m=n}^{\infty} U^m H \subset \bigvee_{m=1}^{\infty} U^m H,$$

we conclude that

$$TPf = PUf \quad \text{for all} \quad f \in R_+. \tag{8}$$

Thus by (5) and (7)

$$TPg = PUg = \varepsilon Pg,$$

and this is the first statement of our theorem.

Repeating the above argument with T^*, U^{-1} and $\bar{\varepsilon}$ in place of T, U and ε, respectively, we get $g \in R_-$, $T^*Pg = \bar{\varepsilon}Pg$, and by (3) and (7), $g \in R_0$. This concludes the proof.

Theorem 2. *If*

$$Th = \varepsilon h \tag{9}$$

for an $h \in H$ *and a complex number* ε, $|\varepsilon| = 1$, *then*

$$UP_0 h = \varepsilon P_0 h, \quad h = \varrho PP_0 h, \quad T^* h = \bar{\varepsilon} h.$$

Proof. Theorem A implies for each $g \in K$

$$(P_- h, g) = \lim_n (U^{-n} T^n h, g) = \lim_n (U^{-n+1} T^{n-1} \varepsilon h, Ug) = (P_- \varepsilon h, Ug).$$

Consequently,

$$UP_- h = \varepsilon P_- h. \tag{10}$$

By Theorem 1 this implies $P_- h \in R_0$, and thus by (3) and the definitions of P_- and P_0,

$$P_0 h = P_- h. \tag{11}$$

This fact and (10) prove the first statement of our theorem. Using again Theorem 1, (10) implies

$$T^* PP_- h = \bar{\varepsilon} PP_- h. \tag{12}$$

Now by (11), Theorem A and (9) we have

$$\varrho(h, P_0 h) = \varrho(h, P_- h) = \varrho \lim_n (h, U^{-n} T^n h) = \varrho \lim_n (U^n h, \varepsilon^n h) =$$

$$= \lim_n (T^n h, \varepsilon^n h) = \lim_n (\varepsilon^n h, \varepsilon^n h) = \|h\|^2,$$

i.e.

$$\varrho(h, P_0 h) = \|h\|^2. \tag{13}$$

Again by (11), Theorem A, and (12),

$$\varrho \|PP_0 h\|^2 = \varrho(P_- h, PP_- h) = \varrho \lim_n (U^{-n} T^n h, PP_- h) = \varrho \lim_n (\varepsilon^n h, U^n PP_- h) =$$

$$= \lim_n (\varepsilon^n h, T^n PP_- h) = \lim_n (\varepsilon^n h, \varepsilon^n PP_- h) = (h, P_- h) = (h, P_0 h).$$

This fact and (13) imply that

$$\varrho \|PP_0 h) = \|h\|. \tag{14}$$

Now by (13) and (14)

$$\|h-\varrho PP_0h\|^2 = \|h\|^2-2\varrho \operatorname{Re}(h, PP_0h)+\varrho^2\|PP_0h\|^2 = 0,$$

and consequently $h=\varrho PP_0h$. This fact, (11), and (12) imply $T^*h=\bar{\varepsilon}h$, and the proof is complete.

In connection with Theorem 2 let us recall that G. Eckstein [2] has proved the following statement. If $T\in\mathscr{C}_\varrho$ for some positive ϱ and if the complex number ε of modulus 1 is an approximate eigenvalue of T, then $\bar{\varepsilon}$ is an approximate eigenvalue of T^* with the same approximate proper vectors.

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